

THE
ARENARIUS
OF
ARCHIMEDES,
TRANSLATED FROM THE GREEK,
WITH
NOTES AND ILLUSTRATIONS.

A. R. E. M. A. R. I. U. S.

A. R. C. H. I. M. P. D. S.

TRANSLATION OF THE

NOTES OF THE

Gal 11 2.9

THE
ARENARIUS
OF
ARCHIMEDES,

TRANSLATED FROM THE GREEK,
WITH
NOTES AND ILLUSTRATIONS.

To which is added,
THE
DISSERTATION
OF
CHRISTOPHER CLAVIUS
ON THE SAME SUBJECT,
FROM THE LATIN.

Πυθαγορίων γὰρ λόγος, σώφω ὃν μὲν εἶναι τὸν εὐροῖα τὸν ἀριθμὸν
δεύτερον δὲ τὰν τὰ ἐνόμαζα θεμέων.

PROCLUS.

L O N D O N:
PRINTED FOR J. JOHNSON, ST. PAUL'S-CHURCH-YARD,
AND MESS. PRINCE AND COOKE, OXFORD.
MDCC LXXXIV.



37.
6. 30.
1.

P R E F A C E.

ARITHMETIC is a science so singularly useful in every branch of knowledge, and so immediately necessary to the common affairs of life, that it must doubtless have derived its origin from the earliest ages of the world. To seek for its inventors, therefore, in the annals of antiquity, would be a fruitless and unprofitable enquiry. Like other arts and sciences, from a low and inconsiderable beginning, it advanced towards perfection by slow and almost imperceptible gradation; whilst in the succession of improvements, the original inventors were forgotten.

Amongst those who are recorded to have made any considerable progress in this science, the Phœnicians appear to be the most ancient; an extensive commerce with other nations, rendering the study of it to them absolutely necessary. But of their inventions or improvements no vestiges

A can

152

can be traced. To Greece alone, as in most other literary productions, recourse must be had for the earliest writings on the art of numbering: amongst which the *Arenarius* of Archimedes, is one of the most curious and useful now extant; as it contains a great improvement of the ancient method of calculation, and the invention of another entirely new.

Arithmetic, indeed, before the time of Archimedes, appears to have been in a very advanced state. The knowledge received from the Phœnicians, Pythagoras is said to have encreased with many speculations; and Euclid, in the 7th, 8th, 9th, and 10th Books of his *Elements*, had given a succinct treatise of arithmetic, with many of the particular properties of numbers. But the extent of calculation, known to these mathematicians, was much too limited for all the purposes of speculative science; myriads, or tens of thousands, being the highest numbers for which they had a name. And as these were sufficiently great to express the quantity of all things necessary
for

for the common purposes of life, they contented themselves with investigating the nature, properties, and relations of the included numbers, without endeavouring to extend their calculations farther. Hence, myriads being the greatest number known, the term was frequently used to signify an indefinitely great multitude; and whatever was inexpressible by myriads, was esteemed innumerable, and frequently termed infinite. As the particles of sand are more in number than can be expressed in myriads, without a confused repetition of the term, they were by some supposed to exceed all possible determinate numbers, and by others to be infinite. But to show the possibility of their being exceeded by numbers, and to convince mankind of the extensive powers of numeration, Archimedes published his *Arenarius*: in which he demonstrated, that not only the sands of the earth, but even a greater quantity of particles than could be contained in the immense sphere of the fixed stars, might be signified by some of those

numbers, which he has there invented terms to express.

This is the only work of the ancients, now extant, in which such great numbers are used. The author, indeed, informs us, that this method of inventing terms to denote the values of numbers which exceeded the bounds of the common calculation by myriads, was delivered in his *Ἀρχαὶ* (*a*), or Principles, a work not now extant; of which, what is given in the *Arenarius*, is only a specimen, inserted, lest the terms which he would be obliged to use, should be unintelligible to those who might not have an opportunity of consulting those Principles. All his other arithmetical works being lost, what farther improvements he made in that science, cannot now be certainly known. But from the *Arenarius*, and the rest of his mathematical works which are extant, arithmetic appears to have been, in his time, in nearly as great a state of per-

(*a*) This work is probably that alluded to by Pappus in his *Math. Coll. Lib. 8.*

P R E F A C E

fection as the ancient method of denoting the values of numbers would permit. The superiority of the common arithmetic of the present age arising, in a great measure, from the invention of the cypher; for want of which the ancients were obliged to make use of such a great variety of characters to denote the different numbers, as were burthensome to the memory, and rendered the solutions of questions very laborious.

To this excellent art of numbering by cyphers, Archimedes has approached nearer than any other of the ancient mathematicians, and may at least be said to have laid the foundation (*b*) for its invention in the *Arenarius*. But this important discovery was reserved for a country, which, after many ages of obscurity, made such proficiency in science, as to instruct Europe, and contend even with Greece for some of the greatest improvements in mathematics. To Arabia we are indebted for the use of the cypher: a country which claims also that excellent improvement in Arithmetic, the

(*b*) See Note on p. 22 *Arenarius*.

invention of Algebra. But to this her claim is not so indisputable as the other (*b*): as the treatise of Diophantus on this subject is much more ancient than any Arabic production now known. Nor is there much doubt, but that this branch of science was in use long before his time. He treats it as a science already known, and which he intended rather to methodize and arrange in order, than to teach its first rudiments: and as several vestiges of it are discoverable in much earlier writers, the invention of Algebra must have been of much greater antiquity. The discovery of the analytic part, is, indeed, by Theo, ascribed to Plato, who defines it to be (*c*) “*Assumptio quæfiti tanquam concessi per consequentia ad verum concessum*: The taking that as granted, which is the quantity sought, and by the consequences thence arising, obtaining the true answer.” Something similar to this is the method used by Archimedes in the demonstration of a propo-

(*b*) See Dr. Wallis’s *Hist. Alg.* and Vossius *De Scien. Math.*

(*c*) See Vieta’s *Ifagoge in Artem Analyticam*, pag. 1. also Diogenes Laertius’ *Life of Plato*.

tion in the following work. He assumes a quantity as known, and by going back, from the conditions of the proposition, he proves, that the value of that assumed quantity is equal to a particular known determinate one. From this and several passages in other parts of his works, it evidently appears, that he made use of some kind of Algebra in the invention and demonstration of many of his propositions. And it seems very probable, that some such method was made use of by earlier mathematicians in the investigation of their more abstruse and difficult problems: since it is hardly conceivable, that the demonstrations given by the ancient writers in geometry, were first discovered in the same elegant manner in which they were left to posterity. But by whatever method these discoveries were made, every necessary instruction is given to make their demonstrations clearly intelligible, without its being known. As they are drawn from simple, undeniable principles, attention only is necessary to make them understood; and when understood, they indubitably evince

their truth. The geometrical method of demonstration being more elegant and perspicuous than any other, was held in the highest esteem by the ancients: they regarded it as the most excellent of sciences, without which, no one could be perfectly master of any kind of learning. Equal to their ideas of its importance, was their assiduity in cultivating and improving it; and as in all other branches of science in which they particularly exercised their genius, so also in geometry their works have ever been esteemed the standard of taste and elegance. But, as every thing in the extreme loses of its excellence, so the veneration of those ancients for geometry being carried too far, diminished the value of their improvements: for its sake alone, and the satisfaction which abstract reasoning affords to the mind, its application to arts equally useful and necessary in the affairs of life, was neglected and despised. To apply mathematical knowledge to mechanical inventions and experiments, was regarded as derogatory

derogatory to the science, and unworthy a geometrician. Archytas (*d*), who first applied geometry to mechanics, met with the reprehension of Plato, who esteemed the mathematical sciences of too sublime a nature to be made common. This opinion in general prevailed; and mechanical and experimental knowledge was almost universally neglected. Archimedes appears to have been one of the first, who after the time of Archytas and Eudoxus, applied the sciences of geometry and mechanics to their mutual illustration, and is equally celebrated for his inventions in the one, as for his extensive knowledge in the other. But he, according to Plutarch, was so far influenced by the then prevalent opinion of Plato, that he would not commit to writing any account of his mechanical inventions (*e*) (though this indeed may be accounted for from other reasons.) Mechanical experiments being thus neglected and despised, no other method of accounting for the various operations of nature re-

(*d*) See Diogenes Laertius, and Ramus in Math. Schol. Lib. 1.

(*e*) Plutarch's Life of Marcellus.

mained,

mained, but that of framing hypotheses ; which, depending only upon reasoning drawn from probability, were defended by the followers of one sect, and rejected and confuted by those of another. Amongst the variety of hypotheses which were framed on different parts of philosophy, chance or reason directed some to the right system of things. Of these was Aristarchus the Samian, who rejecting the vulgarly received opinion of the motion of the heavenly bodies, proposed the true system of the world. Of this philosopher's works, none, except a small treatise on the relative magnitudes of the sun, moon and earth, are now known : and the only account we have of this system is in the *Arenarius* ; which is there too concise, to give any idea by what reasoning he refuted the vulgar opinion, and established his own. The age in which he flourished (*f*) is not perfectly known : and hence have arisen many disputes, whether he was the original inventor of this system, or followed some other.

(*f*) *Blancanus* supposes him to have lived about the 89th olympiad, a little after the birth of *Plato*. *Math. Chron.*
page

other. Archimedes speaks of him as expressly the chief who rejected the vulgar opinion (*g*); Ταῦτα γὰρ ἐν ταῖς γραφομέναις παρὰ τῶν Ἀσρολόγων διακρέσας Ἀρίσταρχος ὁ Σάμιος ὑποθεσιῶν ἐξέδωκεν γράψας. And Sextus Empiricus (*h*) seems of the same opinion. The propagation of this system in Greece is, however, generally ascribed to Pythagoras. But as the time of Aristarchus is uncertain, and as the authorities which attribute this system to Pythagoras, (or the more ancient Egyptians from whom he is said to have received it,) are not so indisputable, but that (*i*) writers of the first eminence have refused their assent to them; there appears at least an equal reason for

page 46 and 49.—Plutarch also speaks of him as following Plato. *Quest. Plat. Phil.* p. 1006.—Simler places Aristarchus about the 79th olympiad.—Vitruvius, speaking of mathematicians who had made discoveries, mentions him before Archytas.—And Libertus Fromondus supposes that he might have preceded even Pythagoras.

(*g*) Arenarius, page 2.

(*h*) *Adversus Mathematicos*, page 410. “ They who
“ taking away all motion from the world, teach the
“ motion of the earth as the followers of Aristarchus the
“ mathematician.”

(*i*) See Dr. Horsley's Notes on Sir Isaac Newton's *Principia*.

termining

terming it the Ariftarchian (*k*), as the Pythagorean fyftem. But whoever was the real inventor, Ariftarchus appears to have entertained the moft juft ideas of its vaft extent: what the obfervations of Dr. Bradley evince, he has fupposed; viz. that the annual orbit of the earth, compared with the diftance of the fixed ftars, is but as a point (*l*).

The reception which Ariftarchus's fyftem met with amongst the Greeks, is equally uncertain with the time of his publishing it. It feems probable, however, that the apparent motions of the heavenly bodies had greater influence over their minds than his reasonings; as the vulgar opinion of the immobility of the earth in general prevailed; this fyftem being totally difregarded for ages, till the time of Copernicus: by whose demonftrations, and thofe of Sir Ifaac Newton, its truth is at length fo fully eftablifhed, that none,

(*k*) Libertus Fromondus terms it the Ariftarchian Syftem. See Ant. Ariftarchus.

(*l*) Arenarius, page 2 and note.

but

but those who cannot comprehend their works, will ever deny its being THE ONLY TRUE SYSTEM OF THE WORLD. Whether Archimedes adopted this opinion or not, cannot be certainly determined; in the following work he speaks neither against it, nor in support of it: And but little mention is made of astronomy in any other of his works which are now extant.

The Arenarius being a work less geometrical than the rest of Archimedes' writings, seems to have been wholly neglected by the old commentators: hence it retains more of the Doric dialect, in which the author wrote, than those which they have transcribed and corrected; but in so mutilated a state thro' the inaccuracy and ignorance of transcribers, that it is often difficult even to obtain the real signification. This has engaged several modern mathematicians to correct the text, and explain its obscurities: and if the following translation shall be deemed a farther elucidation, it is owing to their united helps, and
the

the comparing of the different editions. Of these the most ancient is one printed by Hervagius (*m*) at the Basilean press, A. D. 1544, as it seems, under the direction of Thomas Gechauff, from some inaccurate and, in many places, obliterated manuscript, without any corrections being offered. But mutilated as it is, the sense may in most cases be obtained; and as it preserves the genuine reading of some ancient MS. it is a valuable edition. The next is one published by Rivaltus, A. D. 1615, from the Parisian press. This editor has attempted to correct the text; but his alterations are rather the result of his own conjectures, than of any collation of different copies, and are not always happily introduced. Dr. Wallis also published an edition of this work, in which he has corrected the text according to the Doric dialect, and given the different readings of preceding editors; which makes his much superior to any other. There is also an anonymous edition, without any title, date, or

(*m*) In the Bodleian Library at Oxford is this edition, with manuscript corrections by Dr. Bernard.

place

place of printing, which differs in many places essentially from all the others, some of which are mentioned in the notes. This is a small folio, containing the Arenarius, and a (n) Theorem from Ptolemy's Almagest prefixed; which is there said to have been used by Archimedes in the Arenarius. The Greek of this edition seems in general very correct; and the character is modern and elegant.

Of the Latin translations (o) Hervagius's, which is published together with his Greek edition, appears to be the most ancient: but from the mutilations of the MS. he

(n) The following is the Theorem alluded to. "If two unequal right lines be drawn in a circle; the part of the circumference cut off by the greater, shall bear to the part cut off by the less, a greater ratio than the greater of the two lines themselves bears to the less." The only part of the Arenarius where this could be of any service, is in the demonstration of the proposition, page 14, which the author has not given, and which is demonstrated without it in the notes. But Commandine and Barrow in their demonstrations, have referred to this Theorem.

(o) This seems to have been written long before the publication of the Greek text, and is mentioned in Gechauff's preface as the work of Jacobus Cremonensis. The above-mentioned anonymous edition of the Arenarius, seems to have been altered on purpose to agree with this translation.

used,

used, the translator has, not infrequently, differed from the real meaning of the author. Another is by Paschasius Hamellius; printed at Paris with a large commentary, A.D. 1557, which is very different from the former, and is only excelled in correctness by Dr. Wallis's. Commandine has also given a translation, with notes, of this treatise, but thro' the inaccuracy of his copy (of which he much complains) he is often erroneous. Together with his Greek edition, Rivaltus published the Latin with notes: but as his corrections of the Greek text cannot in all places be defended, so neither can his translations. Besides these there are several others to be met with; but they seem to be rather copies of some of the above, than any original attempts to elucidate the author, except Dr. Wallis's and Dr. Barrow's; the former of which is accommodated to his own corrected Greek edition; the latter may more properly be called an abridgement than a translation of the Arenarius.

Wadham College,
Oxford, 1784.

G. ANDERSON.

THE

THE
ARENARIUS OF ARCHIMEDES,

ADDRESSED TO

PRINCE GELO.

See Note.

THE particles of sand contained in See Note.
Syracuse, the rest of Sicily, and all
other parts of the earth, are by the gene-
rality of mankind esteemed innumerable;
some affirming that they are absolutely
infinite, and others, that they exceed the
sum of any known determinate num-
bers. If, therefore, a mass of sand equal
to the magnitude of this terraqueous globe,
its seas and cavities being filled to the al-
titude of the highest mountains, were pro-
posed, the maintainers of the above opi-
nions would evidently be much more in-
clined

B

See Note. clined to affirm the impossibility of obtaining numbers to express the multitude of particles contained in such a mass. But I shall endeavour to prove by geometrical demonstrations, of whose truth you shall be convinced, that some of those numbers which I have invented terms to express in my writings to Zeuxippus, not only exceed the multitude of sands which would compose the mass above-mentioned, but would even express more particles

See Note. than the whole world itself could contain. And the term WORLD, as it is defined by most astronomers, is here designed to signify a sphere of the heavens, whose center coincides with the center of the earth, and whose semidiameter is the distance from the center of the earth to the center of the sun. This definition of the term WORLD, as given in the writings of other astronomers, Aristarchus, the Samian, has refuted, and given it a far more extensive signification: For, according to his hypotheses, neither the fixed stars nor the sun are subject to any motion; but the earth annually revolves round the sun

See Preface.

sun in the circumference of a circle, in the center of which the sun remains fixed; and the sphere of the fixed stars, whose center he supposes to coincide with the sun's, is of such immense magnitude, that the circle, in whose periphery the earth is supposed to revolve round the sun, bears no greater proportion to the distance of the fixed stars, than the center of a sphere does to its superficies. This is, however, evidently absurd; for, since the center of a sphere possesses no magnitude, it can bear no proportion to the superficies: Taking therefore the earth as the center of the world, let Aristarchus be supposed to signify by this hypothesis, "That the earth bears the same proportion to the sphere of the world, (according to the common definition of the term) as the sphere, in a great circle of which he supposes the earth to revolve, bears to the sphere of the fixed stars." For the demonstrations of the phænomena of the heavens, which he has given, seem particularly to agree with this hypothesis; and from what has been said it is evident, that

See Note.

See Note.

the sphere in which he supposes the earth to revolve, is the same with that which is commonly termed the World.

See Preface.

Now, if a mass of sand, equal to the magnitude assigned by Aristarchus to the sphere of the fixed stars, were proposed, I affirm that some of the numbers, to express the values of which terms are invented in my Principles, would even exceed the number of particles which such a mass could contain.

See Note.

In order to demonstrate which, I shall premise the following propositions:

See Note.

PROP. I. The circumference of the earth is not greater than three hundred myriads (three millions) of stadia.

For since some have attempted to give demonstrations, of the truth of which you seem convinced, that its circumference is nearly three hundred thousand stadia, by making it ten times as great, I shall certainly exceed the true circumference;
and

and therefore shall make use of three millions as its circumference, but not more.

PROP. II. The diameter of the earth is greater than the diameter of the moon; and the diameter of the sun greater than the diameter of the earth.

For most of our ancient astronomers are of this opinion.

PROP. III. The sun's diameter is not greater than thirty times the moon's. See Note.

For, amongst the ancient astronomers, Eudoxus has shown it to be about nine times; Phidias, the son of Acupater, about twelve; and Aristarchus has attempted to demonstrate, that the sun's diameter is greater than eighteen, but less than twenty times the moon's; therefore, by supposing it about thirty times, I shall exceed all others, and consequently render my demonstrations incontrovertible; for which reason, throughout this work, I

shall estimate the sun's diameter at thirty times the moon's, but not greater.

PROP. IV. The diameter of the sun exceeds the side of a chiliagon (or thousand-sided figure) inscribed in a great circle of the sphere of the world.

See Note. This I propose, because Aristarchus has affirmed, that the sun appears to occupy about the seven hundred and twentieth part of a great circle of the zodiac.

See Note. Considering in what manner to determine this, I endeavoured, with an instrument, to observe the angle subtended by the sun's diameter, to the eye of an observer on the earth: To take which exactly is no easy matter, as neither our eyes, hands, or the instruments which must be used, are sufficiently free from error, to make observations with accuracy. But complaints of this kind, are so frequently to be met with elsewhere, as renders it quite unnecessary for me to dwell on them in this place. Since, for the demon-

demonstration of this proposition, it is sufficient for me to take an angle near to that subtended by the sun's diameter, and which is certainly not less; and also another angle near the same, and which is certainly not greater than the angle subtended by the sun's diameter to an observer on the earth; which may be obtained in the following manner:

On a perpendicular fulcrum, erected in a place where the rising of the sun may be observed, let a long rod be placed, and upon this rod, in a perpendicular situation, a small cylinder, made perfectly circular. Then immediately after the sun is risen, just as it becomes wholly visible in the horizon, let one end of the rod be directed towards it, and the eye applied to the opposite end, with the cylinder so near it, as to prevent any part of the sun's body from being seen; the cylinder must then be removed from the eye till a small part of the sun's body becomes visible on each side of it, and there fixed. Now if the perceptive part of the eye were of no sensible

See Note.

fible magnitude, the rays emerging from the body of the sun on each side of the cylinder would converge to a point on their entering the eye; and therefore, if right lines were drawn from the extremity of the rod, where the eye was placed, to touch both sides of the cylinder, in those parts where the sun's body just became visible, the angle contained by those lines would be nearly equal to that subtended by the sun's diameter, but something less, because some part of the sun's body was observed on each side of the cylinder. But

See Note. the pupil of the eye, through which the rays of light enter, being of a sensible magnitude, it is necessary, before an angle something less than that subtended by the sun's diameter can be taken with any degree of exactness, to determine what this magnitude is. In order to which, let two small cylindrical bodies, of equal diameters, be taken, the one white, the other not, and placed in a right line before the eye, the white one at some distance, and the other as near as possible, so as to touch the face. Now if those cylindrical bodies
be

be taken less than the pupil of the eye, some part of the pupil will not be covered by that which is nearest it, and the white one will be seen on each side of the other; in part only if their diameters be but little less than the diameter of the pupil; but if they are much less, that next the face will be entirely absorbed by the rays of light which fall on the eye, and the whole of the white one will become visible. Having therefore, by repeated trials, obtained those cylindrical bodies of such a magnitude as that the one shall exactly cover the other by its shadow, and not more; their diameters, as is evident, will nearly determine the diameter of the perceptive part of the eye; however, they cannot possibly be less. Therefore if a circular body of this magnitude, so determined, be placed at the end of the rod, in the place where the eye was fixed during the time of making the observation, and right lines be drawn touching the sides both of the cylinder and this circular body, they will converge to a point at a small distance beyond the extremity of the rod, and there form

form an angle, evidently something less than that subtended by the sun's diameter.

To take an angle near to that subtended by the sun's diameter, and which is certainly not less, let the cylinder be moved to such a distance from the eye, as just to cover the whole body of the sun and prevent its being seen; and then lines drawn from the place of the eye, to touch each side of the cylinder, will evidently contain an angle something greater than what the sun's diameter subtends to an observer on the earth.

Having determined those angles in the above manner, and reduced them to parts of a right angle, the greater will be found to be less than the hundred and sixty-fourth part of a right angle, and the less greater than the two hundredth part of a right angle; consequently the sun's diameter subtends an angle less than the hundred and sixty-fourth, and greater than the two hundredth part of a right angle.

Which

Which being premised, it will be easy to demonstrate, That the sun's diameter subtends a greater arc of a great circle of the sphere of the World, than the side of a Chiliagon inscribed in the same: For suppose a plane to pass through the center of the earth and the observer's eye (the sun being supposed, as before, just risen above the horizon) which produced shall cut the sphere of the World in the circle ABG , the earth in the circle DEZ , and the sun in the circle SH : Let F represent the center of the earth, K of the sun, and D the place of the observer on the earth: From D let the right lines DL and DX be drawn, touching the circle SH in the points N and T ; and from F , FM and FO touching the same circle in the points R and P , and cutting the circle ABG in the points A , B ; and let DK and FK be joined. Then the sun being supposed above the horizon, DK is consequently less than FK , and therefore the angle $XD L$, contained by the tangent lines DL and DX , is greater than the angle OFM contained by the tangent lines FO and

Fig. I.

and $F M$. But $X D L$ is the angle subtended by the sun's diameter to an observer on the earth, and is therefore greater than the two hundredth part of a right angle, and less than the hundred and sixty-fourth, as is manifest, from what has been

Fig. I. said above. The angle $O F M$, being therefore less than $X D L$, is less than the hundred and sixty-fourth part of a right angle, and consequently the right line $A B$, subtending this angle, cuts off a segment of the circle $A K B$ less than the hundred and sixty-fourth part of a quadrant, or (which is the same thing) less than the six hundred and fifty-sixth part of the whole circumference. Now (as I have demonstrated in another work) the circumference of every circle being to its diameter in a less ratio than twenty-two to seven, therefore, the perimeter of every inscribed polygon (being less than the circumference of the circle) must be to the semidiameter in a less ratio than forty-four to seven. The perimeter of the polygon therefore, of which $A B$ is one side, bears a less ratio to $F K$ (the semidiameter) than forty-four

four to seven: and AB (subtending less than the six hundred and fifty-sixth part of the whole circumference) is therefore to FK in a less ratio than eleven to one thousand one hundred and forty-eight, and consequently is less than the hundredth part of FK . But AB is equal to the diameter of the circle SH : For let RK be joined, then because FA touches the circle SH in R , RK is perpendicular to FA ; and because FK bisects AB at right angles in C , and is equal to FA , (being both radii of the same circle) therefore the triangles FKR and FAC are both right-angled, and have the angle AFK common, and the sides FK and FA equal, and consequently the two sides CA and KR must also be equal: But KR is the semidiameter of the circle SH ; CA is therefore equal to its semidiameter, and consequently AB , the double of CA , is equal to the diameter: Hence the diameter of the circle SH is less than the hundredth part of FK ; and since the circle SH is greater than the circle DEZ , the diameter EU of the circle DEZ must be much less than the hundredth part of FK .

Fig. I.

FK. Therefore FU and KS, the semidiameters of those circles, taken together, are less than the hundredth part of FK: and consequently by taking them both from FK, we have the ratio of FK to US less than that of one hundred to ninety-nine. And because FP is less than FK (FK subtending the right angle at P) and SU than

Fig. I. DT (because the shortest distance between two circles is in the right line joining their centers) the ratio of FP to DT must therefore be less than one hundred to ninety-nine. Now the triangles FKP and DKT being rectangular, and the sides KP and KT equal to each other, and FP and DT unequal, therefore the greater angle KDT, contained by the sides KD and DT, is to the less angle KFP, contained by the sides FP and FK, in a greater ratio than FK to DK,

See Note. but in a less than FP to DT.—For, if two right-angled triangles have one of the sides about the right-angle equal in both, and the other sides unequal; the angle contained by the hypotenuse and unequal side of the less triangle, bears to the angle

contained by the hypotenuse and unequal side of the greater, a greater ratio than the hypotenuse of the greater does to the hypotenuse of the less; but a less ratio than the unequal side of the greater about the right angle bears to the unequal side about the right angle of the less.—Therefore the angle TDN (the double of KDT) is to the angle PFR (the double of KFP) in a less ratio than FP to DT: that is, (from what was proved above) in a less ratio than one hundred to ninety-nine. But TDN or XDL is the angle subtended by the sun's diameter, and is therefore greater than the two hundredth part of a right angle; consequently PFR, being greater than ninety-nine hundredth parts of the angle TDN, must be greater than ninety-nine twenty thousandth parts of a right angle, which is nearly one two hundred and third part of a right angle, or one eight hundred and twelfth part of four right angles, or of a complete circle. Therefore BA, which subtends this angle PFR, must be greater than the subtense of the eight hundred and twelfth part of the

the

Fig. I.

the circumference of a circle, or (which is the same thing) greater than one side of a polygon of eight hundred and twelve sides inscribed in a circle. Hence therefore it evidently follows, that the sun's diameter, represented by A B, is much greater than the side of a Chiliagon, or thousand-sided figure, inscribed in a great circle of the sphere of the world.

See Note.

PROP. V. The diameter of the sphere of the world is less than ten thousand times the diameter of the earth, or less than ten thousand millions of stadia.

For since the sun's diameter is estimated at less than thirty times the moon's, and the moon's diameter is less than the earth's; therefore the sun's diameter must evidently be estimated at less than thirty times the earth's : And again, because the sun's diameter is greater than the side of a Chiliagon inscribed in a great circle of the sphere of the world, the perimeter of such a Chiliagon must consequently be less than a thousand times the sun's diameter, or (as the

the sun's diameter is estimated at less than thirty times the earth's) less than thirty thousand diameters of the earth. Hence (the diameter of every circle being less than a third part of the perimeter of every polygon of more than six sides inscribed in it) it follows, that the diameter of the sphere of the world is less than ten thousand diameters of the earth; and since the circumference of the earth was estimated to be not greater than three hundred myriads (three millions) of stadia, its diameter must therefore be less than one hundred myriads (a million,) and consequently the diameter of the sphere of the world being less than ten thousand times this, must be less than an hundred myriads of myriads, or ten thousand millions of stadia.

Having premised thus much concerning magnitudes and distances, I should now proceed in my principal design of demonstrating the possibility of expressing by numbers a multitude greater than the particles of sand: But lest the terms I shall make use of should be unintelligible to

See Note.

C

those

those who have not had an opportunity of consulting my writings to Zeuxippus, it will not perhaps be thought improper to premise, first, something concerning the method of inventing terms to express the values of numbers greater than the common method of calculation affords; which may be done in the following manner:

Tradition has delivered to us terms which express the values of numbers as great as a myriad, (10,000) and, by proceeding in the same manner as before we arrived at a myriad, we easily calculate to myriads of myriads (100,000,000): And, to continue our calculation farther, let all the numbers included by myriads of myriads be termed the first degree of myriads; and let the myriads of myriads of the first degree be termed the units of the second degree of myriads; and let the units of this second degree of myriads be calculated by tens, hundreds, thousands and myriads, to myriads of myriads, which let again be termed units of the third degree of myriads; and having in like manner calculated

lated the units of the third degree, to myriads of myriads, let these be also termed units of the fourth degree of myriads; and, proceeding in the same manner, we calculate to the fifth, sixth, seventh, &c. degrees of myriads; and by this method terms may be obtained to express the values of numbers as great as myriads of myriads of the myriomyresimal degree of myriads. Which terms comprehend a multitude sufficiently great for our present purpose.

This method of inventing terms, may, however, be extended to a much greater degree. For let the numbers, to express which terms have been invented above, be called the first period, and the last number of the first period, the units of the first degree of myriads of the second period, and, again, let myriads of myriads of the first degree of the second period, be termed the units of the second period of the second degree of myriads; in like manner let the last of this denomination be termed the units of the second period of the third degree of myriads, &c. By proceeding

ceeding in this manner, we can easily acquire terms to express the values of numbers as great as myriads of myriads of the myriomyrefimal degree of myriads of the second period : Which may, to extend our calculation farther, be termed the units of the third period of the first degree of myriads. And hence, by proceeding in this manner with the periods, terms are obtained to express the values of numbers to myriads of myriads of the myriomyrefimal period of the myriomyrefimal degree of myriads.

See Note.

Terms being by this method invented to express the values of numbers however great, an easy method of determining to what particular denomination any expression belongs, may be drawn from considering the relation between the denomination of numbers and the terms of a series of continued proportionals. For supposing a series of numbers continually proportional from unity, to increase in a ten-fold ratio, the first eight terms of such a series, together with unity, (the first term) will

will constitute those numbers which are denominated the first degree of myriads. And also the eight terms following these, will be comprehended in the second degree of myriads: and in the same manner all the following eight terms in such a series of proportionals will be the component parts of a new expression, in the above denomination of numbers, which is distinguished from the preceding numbers by the number of eighth terms it is distant from unity. Thus the eighth term of the first eight proportionals, in such a series, is thousands of myriads, and the next text term following, (the first of the second eighth division) being ten times the preceding, is myriads of myriads; or, according to the denomination above assigned, the units of the second degree of myriads: Again, the following eighth term, in the same series, is thousands of myriads of the second degree of myriads; and therefore the next term following, which is the first of the third eighth division, being ten times as great, is myriads of myriads of the second degree, and consequently units of the third

See Note.

degree of myriads. From which it is evident, that the several different denominations by which numbers are distinguished from each other, in the method above delivered, exactly correspond to the several eighth places from unity in such a series of continual proportionals. Now, in any series of continual proportionals, whose first term is unity, if any two terms be multiplied into each other, the product, thence arising, will be a proportional term in the same series, as many terms distant from the greater of the two which were multiplied together, as the less is distant from unity: And the number of terms it is distant from unity, may be determined by the addition of the distances from unity of the two terms so multiplied together, and subtracting an unit from the sum. For let there be any series of continual proportionals $a b c d e f g h i k l$, the first term a of which is unity, and let any one term of the series, as d , be multiplied by another h , and let their product be termed x ; also let l be another proportional in the same series, as many terms distant from

from b , the greater, as d , the less, is from unity, then l is equal to x . For since the series is continually proportional, and d is distant as many terms from a as l is from b , therefore b has the same ratio to l that a has to d . But a being unity, d is equal to a multiple of it by d ($=da$), and consequently l must be equal to a multiple of b by d ($=db$), or equal to x . Wherefore the product arising from the multiplication of any two terms in a series of continual proportionals from unity, is a term in the same series, and is as many terms distant from the greater of the two so multiplied, as the less is distant from unity. And hence, by adding those two distances together, and subtracting unity from the sum, the number of terms which the product is distant from unity is obtained: For, as the distance of d from a includes d , and the distance of b from unity includes d also, therefore the distance of l from b , being as many terms as d from a , is one less than the sum of both the distances of d from a , and b from a ; consequently, if from the sum of those two

distances unity be subtracted, the remainder will give the place of the product l in the same continued series of proportionals.

Now, having premised and demonstrated every thing necessary to render the work intelligible, I next proceed to demonstrate the possibility of expressing, by numbers, the quantity of particles contained in any mass of sand whatever. In order to which, I shall take it for granted, that a small
See Note. spherule of the bigness of a poppy-seed could not contain more than ten thousand particles of sand; and that the diameter of such a poppy-seed is not less than the fortieth part of an inch: Indeed it is much greater, for by laying a number of poppy-seeds on a plane, in a right-line touching each other, I found that twenty-five occupied a longer space than the extent of an inch: But that what is delivered on this subject may be demonstrated beyond a possibility of doubt, I shall suppose them much less, and therefore estimate the diameter of one seed at the fortieth part of an inch,

inch. And since a sphere of this magnitude is proposed to contain less than ten thousand, or a myriad of particles of sand, therefore, spheres being to each other in the triplicate ratio of their diameters, a sphere of forty times this diameter, or one inch, would contain a less number of particles than the product arising from multiplying ten thousand by sixty-four thousand, or one myriad multiplied by six myriads four thousand; which, according to the denomination assigned above to great numbers, makes six units of the second degree of myriads, and four thousand myriads of the first.

A sphere of sand, therefore, of an inch in diameter, would contain a number of particles less than six units of the second degree of myriads, and four thousand myriads of the first degree: Which is evidently much less than ten units of the second degree of myriads. And, therefore, a sphere of an hundred inches diameter, being an hundred myriads of times the magnitude of that which is but one inch,

inch, will contain a less number of particles than is the product of ten units of the second degree of myriads multiplied by an hundred myriads. But ten units of the second degree of myriads, constitutes the tenth term in a series of proportionals continually increasing in a ten-fold ratio from unity; and an hundred myriads the seventh from unity in the same series; consequently the product arising from the mutual multiplication of these two will (by the property of continual proportionals above demonstrated) be the sixteenth proportional term from unity in the same series; that is, one less than the sum of ten and seven,

Of these sixteen proportional terms, the first eight, with unity, constitute the first degree of myriads; and consequently, the eight following belong to the second degree, and therefore make thousands of myriads of the second degree of myriads: So that a thousand myriads of the second degree of myriads of particles, would more than compose a sphere of sand of an hundred

dred inches in diameter. Again, a sphere of a myriad of inches in diameter, is an hundred myriads of times the magnitude of one of an hundred inches in diameter; and therefore by multiplying the above quantity of particles, which are more than could be contained in a sphere of an hundred inches in diameter, by an hundred myriads, the product will evidently exceed the quantity contained in a sphere of a myriad of inches in diameter; or by adding to the sixteenth place last found, the place occupied by hundreds of myriads in a series of continued proportionals increasing in a ten-fold ratio, which is the seventh, and subtracting one, the place occupied by the product in the same series will be obtained; which is therefore the twenty-second from unity in the same series, and consequently is equal to ten myriads of the third degree of myriads. But this sphere of a myriad of inches in diameter, is greater than one of a stadium in diameter; consequently a sphere whose diameter is equal to a stadium, might be composed of less particles than ten myriads of the third degree

gree of myriads. In like manner by multiplying this last number by an hundred myriads, a number is obtained greater than the multitude of particles of sand requisite to compose a sphere of an hundred stadia in diameter; which number will be found, by proceeding as in the last instance, to occupy the twenty-eighth place from unity in the series of proportionals, which is equal to a thousand myriads of the fourth degree of myriads. Proceeding thus by multiplying continually by an hundred myriads, or, which is the same thing, by adding to the proportional term last found seven, and subtracting unity from the sum, a number greater than the multitude of sands which could be contained in a sphere of a myriad of stadia in diameter, is found to be ten units of the fifth degree of myriads: And in a sphere of an hundred myriads of stadia in diameter, the number of particles it could contain, is found to be less than a thousand myriads of the fifth degree: Also in a sphere of a myriad of myriads of stadia in diameter, the number of particles is less than ten myriads of

of the sixth degree of myriads: And in a sphere of an hundred myriads of myriads of stadia in diameter, the number of particles it could contain is thus proved to be less than the fifty-second term of the above-mentioned series of proportionals, continually increasing in a ten-fold ratio, or less than a thousand of the seventh degree of myriads. And since (by Prop. V.) a sphere of an hundred myriads of myriads of stadia in diameter, is greater than the sphere of the world, therefore the number of particles of sand which could be contained in the whole world, according to the magnitude assigned to it by most astronomers, is less than a thousand of the seventh degree of myriads.

From which it may also be easily shown, that the number of sands, which could be contained in a sphere equal to that which Aristarchus has assigned to the fixed stars, is less than a thousand myriads of the eighth degree of myriads.

For

For since the earth is supposed to have the same ratio to the sphere of the world (according to the commonly received signification of the term) as the same sphere of the world has to the Aristarchian sphere of the fixed stars, therefore the diameters of those spheres have the same ratio to each other; and because the diameter of the sphere of the world is (by Prop. V.) less than a myriad of times the diameter of the earth, therefore the diameter of the sphere of the fixed stars is less than a myriad of times the diameter of the sphere of the world. Hence, spheres being to each other in the triplicate ratio of their diameters, the sphere of the fixed stars (according to the magnitude assigned by Aristarchus) is less than a myriad of myriads of myriads of times the magnitude of the sphere of the world. But the number of sands which could be contained in the world, has been demonstrated to be less than a thousand of the seventh degree of myriads; multiplying therefore this number by a myriad of myriads of myriads, a number will be obtained, evidently greater than

than the multitude of particles of sand which could be contained in a sphere equal to the Aristarchian sphere of the fixed stars. But a thousand of the seventh degree of myriads, occupies the fifty-second place in the above-mentioned series of proportionals, and a myriad of myriads of myriads the thirteenth; therefore the product arising from the multiplication of those two terms, will be the sixty-fourth term in the same series from unity; that is, the eighth place from the eighth degree of myriads, or a thousand myriads of the eighth degree of myriads. A sphere of sand, therefore, equal to the Aristarchian sphere of the fixed stars, might be composed of a less number of particles than a thousand myriads of the eighth degree of myriads. See Note.

These things, Prince Gelo, I apprehend will to most people, especially to those who are unacquainted with mathematics, appear entirely incredible: But to those who have made this science their study, and

and have attentively considered the respective distances and magnitudes of the earth, sun, and moon, and the sphere of the whole world, the demonstrations here given will make them clearly evident. For which reason I am of opinion, that studies of this nature ought to be much more attended to.

NOTES

N O T E S

A N D

I L L U S T R A T I O N S.

Page I.—Gelo.] **T**HIS prince was the eldest son of Hiero, king of Sicily, a relation of Archimedes. He is supposed to have fallen a sacrifice to his father's fidelity to the Roman state, his death happening while he was persuading the Syracusians, against Hiero's consent, to revolt from the Romans to the Carthaginians, who, having obtained the victory at Cannæ, appeared to have, at that time, the superior power. The Arenarius must therefore have been written more than two hundred and fifteen years before Christ.

Page I.—Particles of sand, &c.] The Latin translation in the Basilean edition gives a very different signification to this passage. “Ego autem dico non eam
 “ (*arenam*) solum quæ circa Syracusas et circa reli-
 “ quam Siciliam existit, verum etiam illam quæ in
 “ universa habitabili simul et inhabitabili regione ubi-
 “ que continetur, certo quodam numero comprehen-
 “ sam esse.” With this the anonymous Greek edition of the Arenarius alone agrees, having instead of *λεγω, εγω,*
 D and

and adding, to render the sentence complete, *πεπερασμενον φامي ομειν*. But, as there appears no other authority for so material an alteration in the Greek text, and its signification; that meaning is here given which seemed to agree best with the more ancient Greek copies, and which is also given in the Latin translations of Paschasius, Commandine, Rivaltus, and Dr. Wallis.

Page 2.—Much more inclined to affirm.] *Πολλαπλασιως μεν ηγωνισο*, in Dr. Wallis's edition—*μηγουσιν* in the Basilean. In the anonymous edition *πολλαπλασιως* is connected with the preceding sentence, and according to that construction Commandine seems to have read it; as he adds, "Atque hujus ipsius rursus alterum multiplicem." But as Hervagius, Paschasius, Rivaltus, and Dr. Wallis, have connected it with the following verb, their authority is followed in this translation: Though this seems to be a singular instance of the adverb *πολλαπλασιως* being joined with a verb in such a signification; nor does the construction seem more natural, if placed in the manner Commandine has it.

Page 2.—World.] *Κοσμος*; Of this term, Aristotle, in his Tract De Mundo, gives the following definition: "Kosmos is the system of heaven and earth, and of the works of nature contained in them." Agreeable to this is the idea conveyed by, SYSTEM OF THE WORLD; this term is therefore every where used to express the signification of Kosmos. Kosmos is, however, throughout this work, used in a very limited sense: The astronomers who preceded Aristarchus, confined its signification to the orbit apparently described

scribed by the sun in his annual course round the earth. This definition of Kosmos seems to have been given by those, who, because the eye refers all the celestial bodies to the same concave arc of the heavens, asserted that there was but one heaven, (as they termed it) or that the sun and stars were all at the same distance from the earth; and consequently, the path described by the sun in its annual orbit would, in this case, be the utmost limit of the system of nature.

Page 3.—Center of a sphere to its superficies.] This signifies no other than that the annual orbit of the earth, is but as a point compared with the distance of the fixed stars. And that this is really the case, is sufficiently evinced by the observations of Dr. Bradley, which prove, that the angle subtended by the diameter of the earth's orbit, at the distance of the fixed stars, is imperceivably small; and consequently its ratio to that distance, cannot possibly be determined. However, as neither of these spaces are infinitely great, or infinitely small, Archimedes argues that there must be a greater relation between them than between the center of a sphere and its superficies, or between a mathematical point and a sensible magnitude. He therefore changes this analogy for one in which determinate magnitudes alone are concerned; which conjecture of his, is in some measure corroborated by the second position of Aristarchus, in his work concerning the magnitudes and distances of the sun and moon; where he supposes the earth to bear the ratio of a point or center to the sphere of the moon; and consequently, it must bear a less ratio (if it were possible) to the sphere of the solar orbit or world. The earth, there-

fore, bearing an equally small ratio to the sphere of the world, as the world does to the sphere of the fixed stars, Archimedes supposes them to be the same, and from determining the one, the other becomes known. It is not, however, to be supposed, that Aristarchus regarded those ratios as in reality so indefinitely small: But only that they were so small, as that they might be neglected without any sensible error in his calculations.

Page 3.—For the demonstrations.] This passage is very incorrect in all the Greek editions, nor are the Latin translations much more easy to be understood. In the anonymous edition, the construction is the most natural, and seems best to express the author's meaning; and nearly agrees with this translation.

Page 4.—Propositions.] These are rather postulata, or suppositions; the original has only *υποκειμενων των*, “These things being laid down.” Dr. Barrow calls them hypotheses. But as what the author advances, admits of demonstration, and as he has offered some reasons for his assumptions, it seemed not improper to use the term Propositions.

Page 4.—Circumference of the earth.] That the figure of the earth was spherical, appears to have been a very early opinion; and many of the ancient philosophers attempted the measuring of its circumference: And as Archimedes has given the opinion of the philosophers of his time concerning it, it may not be improper to mention the admeasurements assigned by others in preceding and succeeding ages. Aristotle gives 400,000 stadia; Archimedes, 300,000; Eratosthenes, 252,000; Hip-

Hipparchus, 277,000; Ptolemy, 180,000; Alphraganus, an Arabian astronomer, 163,200; Fernelius, a German, 196,114; and some of his followers 152,640, for the circumference of the earth; and if we suppose the stadium to have contained 625 feet, the circumference assigned by modern astronomers will be found to be 211,370. Archimedes, therefore, by supposing it 3,000,000 stadia, used a magnitude whose measure was greater than fourteen times the real circumference of the earth. But, it may here be observed, with regard to these different measurements, that some have attempted to reconcile them to each other, by supposing the stadia to have been of different lengths, in the different times when they were made.

Page 5.—Prop. III.] The sun's diameter being nearly four hundred times the moon's, Archimedes, therefore, though he has given a ratio much greater than that of all other astronomers of those times, has yet supposed it too small; which does not, however, affect his general proposition.

Page 6.—Seven hundred and twentieth, &c.] This seven hundred and twentieth part of the zodiac making $30^{\circ}-00''$ is the same as Kepler and Tycho assign for the magnitude of the sun's least diameter. The following observation of Archimedes tends only to determine the limits between which the angle subtended by the sun is comprehended. The greatest limit which he assigned to the sun's diameter from this observation, is very near the true one: For according to Mayer's tables, the greatest diameter of the sun is nearly $32'-40''$; which differs from Archimedes' one hundred and sixty-fourth

part of a right angle ($32'—55''$) by only fifteen seconds, With respect to the least diameter, indeed, the difference is much greater; Mayer giving $31'—30''$ for the least diameter, and Archimedes one two hundredth part of a right angle, which makes but $27'—00''$ for the least limit. But as in this work no greater exactness was required, than that magnitudes should be taken sufficiently large, Archimedes has made no allowance for the sun being wholly hid by the cylinder, nor for the parts which were visible on each side of it; his purpose only requiring, that the limits assigned should be large enough, and not very far from the truth.. Vague as this observation was, it determined the greatest limit of the sun's diameter nearer the truth than Ptolemy's, as he gives $33'—20''$ for the greatest diameter of the sun.

Page 6, 7.—Instrument.] Plutarch, in his life of Marcellus, mentions Archimedes' having instruments, by which the magnitude of the sun's body might be measured; but of their construction no account is extant. The instrument here described (of which Archimedes himself complains) is so very inaccurate, that it seems strange so celebrated an inventor of machines should make use of it. If the instruments and angles mentioned by Plutarch were an improvement upon this, like most of his other inventions, they were lost; as the same instrument, with very little alteration, was used in the times of Hipparchus and Ptolemy. Of both these instruments Fig. 2 and 3. are representations; the one taken from this description of Archimedes, the other from Proclus; in which, (Fig. 2.) H O represents the horizon; S the sun, just risen above it;

it; F the fulcrum or prop, on which the rod R E is supported; C the cylinder, fixed upon the rod in a perpendicular situation; and E the eye of the observer. The rod being turned towards the sun, and the cylinder so far removed from the eye as that the sun's body shall just begin to become visible on each side of it, the rays D d E and G g E, will appear to touch each side of the cylinder, as they enter the eye, and consequently the angle d E g, formed by drawing lines from E in the same direction with the rays, will be equal to D E G, or the angle subtended by the sun's diameter nearly, &c. (vide Aren, p. 7, 8.) The Greek term *Κατω*, which is here translated Rod, may with equal propriety be supposed to signify any plane surface, as a long ruler, table, board, &c. in which case this instrument differs very little from the Dioptra of Hipparchus, of which Fig. 3. is a representation. In this A B C K represents a plane surface, or table, fixed parallel to the horizon H O, in which a long channel or notch A B is cut, to hold two prisms D E and F G, perpendicular to the surface K C; of these the one F G, is made to move backwards or forwards in the notch A B, and the other fixed at the end B. The moveable one has two holes F and G perforated in it, one near the surface, and the other at some distance from it; and the fixed prism D E has one hole D, for the eye to view any object through in a right line with the hole F of the other. Then, when the sun is risen a little above the horizon, or near setting, the end of the plane surface or table, A is to be directed towards it, and the eye applied to the hole of the fixed prism, D; and the moveable prism D E, moved backwards or forwards in the channel or notch A B,

till the lower part of the sun's body f becomes visible through the hole F , and the upper part g , through the hole G . And therefore by measuring the quantity of the angle $F D G$ equal to $f D g$, the sun's diameter may be nearly determined. This instrument appears to have been made use of by Ptolemy in his observations. Rivalentus has given a different description of Archimedes' instrument, as he supposed that the cylinder lay parallel to the rod, instead of being perpendicular to it.

Page 8.—Pupil of the eye.] The author's method of determining the magnitude of the pupil of the eye, seems to be too fully explained to need any further illustration. But it may not perhaps be improper just to mention a different one, invented some years afterwards by Rabi Levi, as it is in Commandine's note on this passage. In Fig. 4. let AB and DC represent two magnitudes of different lengths, placed in a right line before the eye. Of which the shorter AB is fixed at a given distance EF from the eye E , and the longer DC is to be moved backwards and forwards in the same right line, parallel to the other, and at a greater distance from the eye, till each of its ends D and C just become visible above and below the extremities of the nearer magnitude AB . Then, fixing it in that position, and drawing lines from each of its ends D and C to touch the ends of the other A and B , those lines DAK and CBK will converge to a point, at a something greater distance from those magnitudes than the place of the eye, as at the distance KE , and consequently by drawing the line GH parallel to AB , the distance of the rays of light at entering the eye will be determined.

ILLUSTRATIONS.

41

Page 14.—If two right-angled triangles, &c.] The demonstration of this proposition may be as follows :

Let ADC and BDC , Fig. 5. be two right-angled triangles, having the angle at D right, and the side DC equal and common in both, the sides AD and BD being unequal. The angle DBC is to the angle DAC , in a greater ratio than AC to BC , but less than AD to BD .—Draw CG parallel to DA , and on C as a center, with CB radius, describe the arc BEF , cutting CA in E , and CG in F ; and from B through E draw BEG , meeting CG in G . Also from B draw BH parallel to AC , cutting CD in H , and from B , with BH radius describe the arc HIK , cutting AD produced in I , and BC in K . In the triangles AEB and ECG , the angles at E are vertical and equal; and because GC and AD are parallel, the alternate angles ACG and CAD are equal; therefore, the remaining angles ABE and EGC must be equal. And consequently, the sides about equal angles being proportional, as $BE:EA::GE:EC$, and alternately $BE:EG::EA:EC$. by composition $BG:EG::AC:EC$. But as $BG:EG::\text{triangle } BGC:\text{triangle } ECG$; therefore as $AC:EC::\text{triangle } BGC:\text{triangle } ECG$. Now BE falling within the circle, the sector BCE contains the triangle BCE ; therefore the sector BCE , bears to the triangle EGC , a greater ratio than the triangle BEC does. And since the triangle EGC contains the sector ECF ; therefore the sector BEC , is to the sector ECF , in a greater ratio than to the triangle EGC . And consequently, the sector BEC must have a much greater ratio

ratio to the sector ECF , than the triangle BCE to the triangle ECG . And, by composition, the whole sector BCF , is to sector ECF , in a greater ratio than whole triangle BGC , to triangle ECG . But as sector to sector, so is angle to angle; therefore angle BCF , is to angle ECF , in a greater ratio than triangle BGC : triangle EGC ; that is (from what is proved above) than $AC:BC$. But the angle $BCF = DBC$, and $ECF = CAD$; consequently the angle DBC is to the angle CAD in a greater ratio than AC to BC .

Again, the triangle BHC , contains the sector BHK , and the sector BHI , the triangle BDH ; therefore the triangle BHC is to triangle BDH in a greater ratio than sector BHK : sector BIH : And by composition, triangle BDC , to triangle BDH , greater than sector BKI , to sector BHI : But as triangle BDC , to triangle $BDH :: DC:DH ::$ (by similar triangles) $AD:BD$. Therefore AD is to BD in a greater ratio, than sector IBK to sector IBH ; that is, than angle DBK to angle DBH or DAC , Q.E.D.

Page 16.—Diameter of the World.] Agreeable to the opinions of the astronomers before Aristarchus, the term World (*Koσμoς*) here signifies the apparent annual orbit of the sun round the earth; and, consequently, this diameter must be twice the distance of the centers of the sun and earth. The determination of this distance, has always been esteemed one of the most difficult problems in astronomy. In this method of Archimedes, the magnitude of the sun is supposed to be nearly known; as bearing a less ratio to the earth, than it does to the moon. The manner in which Aristarchus discovered
the

the ratio, given in page 5, was by observing the angle subtended by the semidiameter of the moon's orbit (or distance from the earth) at the sun; from which, by trigonometry, their relative distances from the earth might be easily determinated; and from this their relative magnitudes, by taking the angles subtended by their respective diameters to the eye of the observer. To take this angle subtended by the moon's distance from the earth, at the sun, is, however, attended with the greatest difficulty; as it is necessary, that the exact distance of the moon from the quadrature should be known, when a plane passing from the observer to the moon, would precisely divide its illuminated hemisphere from the non illuminated one: But the moon appears equally enlightened, and in the same direction, for a considerable time, both before and after the exact time of quadrature, and consequently will appear illuminated in the proper direction of such a plane, at very different distances from the place of quadrature; and, therefore, this method was soon found to be impracticable with any tolerable degree of certainty. Nor was any other method of much greater exactness discovered, till the transit of Venus over the sun's disk offered one; from which, in the years 1761 and 1769, the distance of the sun from the earth was determined to be nearly 94,000,000 miles, which is much greater than had ever been supposed before. This distance is, however, far exceeded by Archimedes, where he says, that the diameter of the world, or solar orbit, is less than an hundred myriads of myriads (10,000,000,000) of stadia, since this makes nearly 1,200,000,000 miles, or 600,000,000 for the distance of the centers of the sun and earth, which is more than
fix

fix times their real distance from each other.—Therefore the number of sands, which he has demonstrated would more than compose a sphere of this magnitude, would evidently be more than the whole sphere of the solar orbit could contain.

Page 17.—Having premised.] In the original, the magnitude of a particle of sand (one ten thousandth part of a poppy-seed) is given in this place; which, that what is said of the sand might be all connected together, is inserted at page 23, after the invention of numbers sufficiently large.

Page 20.—Myriads of myriads of the myriomyresimal period of the myriomyresimal degree of myriads.] As it was impossible, in many places of this work, to show the author's method of demonstration and investigation without using some of his own terms, the several values of those terms, according to the present method of notation, are here subjoined,

A myriad is an unit with 4 cyphers affixed	10,000.
A myriad of myriads, or, the first degree of myriads, unity with 8.	100000000
Second, third, &c. degrees, unity with cyphers.	16.24, &c.

The myriomyresimal degree of myriads, or the first period, is unity with 7,9999,9992 cyphers.

A myriad of myriads of the myriomyresimal degree of myriads, or the first degree of myriads of the second period, unity with 8,0000,0000 cyphers.

A myriad of myriads of the myriomyresimal degree of myriads of myriads of the second period, or, the first

first degree of myriads of the third period, unity with 16,0000,0000 cyphers.

The myriomyrefimal degree of myriads, of the myriomyrefimal period, unity with 7,9999,9999,9999,9992 cyphers.

A myriad of myriads of the myriomyrefimal degree of myriads of the myriomyrefimal period, is unity with 8,0000,0000,0000,0000 (eighty thousand millions of millions of) cyphers annexed.

The method by which the invention of terms to express those great numbers is carried on, is very similar to the modern manner of expressing numbers greater than a thousand. For, a myriad of myriads, is termed by Archimedes *μυριας*, or first degree of myriads; and, in the modern arithmetic, a thousand thousands is called millions, or the first degree of thousands: and a myriad of myriads of the *μυριας*, or first degree, *Δεσποιας*, or second degree of myriads; and a thousand thousands of millions, Billions (Bi-millions) or the second degree of thousands, &c.

Page 22.—Now in any series.] Upon this demonstration of the property of continual proportionals, Dr. Wallis founded his assertion that the principles, at least, of the present art of numbering by cyphers were laid down in the *Arenarius*. “For, the numbers designed by the letters a, b, c, d, &c. being in a decuple proportion to each other, are the same with our 1, 10, 100, 1000, &c. and by using those terms, the values of any numbers may be denoted, in a manner similar to that of the present notation; as 1675 in terms of Archimedes’

chimedres' proportion, might be denoted by $1d, 6c, 7b, 5a$, or by using the Greek characters $\alpha d, \epsilon c, \zeta b, \eta a$; and, in like manner, in numbers however large, the same method might be used. Nor are those numbers which he terms $\omega\mu\iota\sigma\tau\omicron\iota\varsigma$, $\Delta\epsilon\upsilon\lambda\epsilon\sigma\iota\varsigma$, &c. (first, second, &c. degrees of myriads) any other than which in our notation respect the first, second, &c. eighth place in a combination of numerical figures.—It may also be further observed, that those terms a, b, c, d , &c. are not only accommodated to the decuple proportion, but to any other that is continual from unity. For, if a be unity, whatever be the value of b and the rest of the proportionals, the same property obtains: And the terms, in every case, exactly correspond to those which are generally denominated unity, root, square, biquadratic, and all the several powers of b in a consequent order. As, let $b=x$, then will $a, b, c, d, e, f, g, h, i, k, l$,

be equal to $1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}$, &c. whatever be the value of b . The invention therefore of characters to denote the several powers of any quantity, commonly called Algebraic or Cossic numbers, is not of so recent a date as some have imagined. And instead of being derived from the Arabians or the later Greeks, is, at least, of as great antiquity as the time of Archimedes, as is evident from the above proportionals."

To this observation of Dr. Wallis's it may not be improper to add, that upon this property is also founded that excellent and useful invention of logarithms, or a series of numbers in arithmetical proportion, corresponding to another in geometrical, by the addition

tion of which, the product arising from the multiplication of their respective natural numbers, is obtained. For, as Archimedes here demonstrates, in any series of continual proportionals from unity, the addition of the distances of any two terms, from the first term, gives the distance of a term in the same series, answering to their product, which from thence becomes known: So logarithms, representing the distance of numbers from unity in some series of continual proportionals, by their addition, the distance of the product, arising from the multiplication of their natural numbers, in the same series, or its logarithm, becomes known, and from that the product itself. The several distances, therefore, of the proportionals $a, b, c, d, \&c.$ are the several logarithms of the values of $a, b, c, d, \&c.$ and are used as such by Archimedes, in calculating the number of sands requisite to compose a sphere equal to the world.

Page 24.—Poppy-feed.] The particles of sand are here supposed so very small, that a single grain would be invisible to the naked eye. For the eye, unassisted, is not able to distinguish so small a portion of space as the five hundredth part of an inch; and according to the number of sands which Archimedes supposes to be contained in a sphere of an inch diameter, one particle could not occupy so much as the one thousand two hundredth part of an inch.

Page 31.—Thousand myriads of the eighth degree of myriads.] That is, one thousand decillions, or the tenth degree of millions; or unity with 63 cyphers annexed.

As

As this number expresses more than the quantity of particles of sand which could be contained in the Aristarchian sphere of the fixed stars, the vast extent to which numeration may be carried is sufficiently proved. And from hence it may be inferred, that no magnitude, however great, has its component parts so small; that the quantity of them is inexpressible by numbers; and this was probably the design of the author in writing this work. In determining the number of particles of sand which would more than compose the several spheres alluded to, Archimedes has used a great number of repetitions to make his method of calculating by proportionals more easy to be understood; and hence the work, in the original, is much more prolix than would otherwise have been necessary: But as this method is now sufficiently known to any one conversant in common arithmetic, those repetitions are as much omitted as seem convenient, without deviating entirely from the author's manner of demonstration. And in other parts of the work, where a trifling deviation from the original seemed to make it more clear and agreeable to an English reader, that liberty has been taken.

THE
DISSERTATION
OF
CHRISTOPHER CLAVIUS,
PROFESSOR OF MATHEMATICS AT ROME,
ON THE
Possibility of numbering the SANDS.
TRANSLATED FROM THE LATIN.

Cum Libro Archimedis de arenæ numero utile fuerit jungere
Digressionem Christophori Clavii de eodem Argumento.
Addenda to Vossius De Scien. Math.

THE UNIVERSITY OF CHICAGO
LIBRARY

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

1000

ADVERTISEMENT.

THE Dissertation of Clavius * on the same subject with the Arenarius, containing the opinions of that eminent mathematician concerning the magnitude of the system of the world; and, expressing in cyphers, according to his calculation what Archimedes has expressed in Greek terms; a translation of it was thought not to be an improper addition to the preceding.

The original is in Clavius's Commentaries on Sacro Bosco's Doctrine of the Sphere; in which (as is necessary to be observed for the better understanding this Dissertation) the Ptolemaic system of the world is every where followed, the Copernican at that time, through prejudice and superstition, being but little countenanced.

Should any one have the curiosity to investigate the same question according to the present admeasurement of the distances of the sun and fixed stars, it may be easily done by proceeding in the same method.

* Clavius flourished about the year 1580.

ns
®

ADVERTISING

The Advertising Agency
has been in the business of
advertising for over 20 years.
We have a large staff of
experienced men and women
who are capable of handling
any advertising campaign.
We have a large stock of
advertising material and
are able to produce any
kind of advertising material
at a very low cost.
We have a large stock of
advertising material and
are able to produce any
kind of advertising material
at a very low cost.
We have a large stock of
advertising material and
are able to produce any
kind of advertising material
at a very low cost.

T H E

D I S S E R T A T I O N, &c.

ARCHIMEDES in the beginning of his Arenarius informs us that, in his time, the particles of sand in Syracuse, and the rest of Sicily, and also those which are in every other part of the earth, were generally supposed to be infinite: and that those who did not affirm their infinity, because an infinite quantity cannot be given, at least, asserted that no given number could exceed their multitude or be equal to it; but rather on the contrary that every proposed determinate number would be far exceeded by the multitude of sands. From which he infers, that those who are of this opinion, if a mass of sand (*a*) many times greater than the magnitude of the

(*a*) See Arenarius, page 1, and note. Clavius here follows Commandine's translation; connecting *πολλαπλασίως* in the original with the preceding sentence.

whole earth, its seas and cavities being filled to the altitude of the highest mountains, were proposed, would without doubt assert, that the quantity of particles contained in such a mass, would far surpass the utmost extent of all the known powers of numeration.

This erroneous opinion Archimedes in his *Arenarius* has, with the greatest ingenuity, by geometrical reasoning, refuted; and investigated numbers which not only exceed the multitude of sands necessary to compose the mass above-mentioned, but even of those which the whole system of the world, supposing it much larger than it really is, could contain. This is the subject of the *Arenarius*, in which, by subtle reasoning, he first demonstrates how to determine the distance of the sun from the earth, by finding an angle near to that subtended by the sun's diameter, &c. (c)

Therefore, following the example of Archimedes, we shall endeavour to deter-

(c) See *Arenarius*, page 7, 8, &c.

mine the number of sands of the smallest magnitude, which would be sufficient to fill the whole space comprehended by the firmament (*d*) of the heavens. And to make this disquisition more evident and worthy of admiration, we shall suppose, that the firmament is of much greater extent than it is found to be by astronomers; and, that the particles of sand are much less than any that are known. For, if we demonstrate the possibility of expressing the number of smaller sands than are ever seen, which would fill a space greater than that comprehended by the firmament, it is evident, that the same number is much greater than the number of the smallest particles of sand in nature, which could be comprehended in that extension of space which astronomers have assigned to the firmament.

In order to determine this we shall premise;

(*d*) The place of the fixed stars according to the Ptolemaic system.

E 4

I. That

I. That the diameter of the earth is much less than 10,000 (*d*) miles. This will certainly be allowed; since according to Ptolemy and other astronomers, its diameter is not greater than $7159\frac{1}{11}$ miles; yet to render our work more easy, and the magnitude of the world greater, we shall suppose it 10,000 miles.

II. That the diameter of the concave firmament is much less than 100,000 diameters of the earth: Indeed according to Alfraganus its diameter is but about 45,225 diameters of the earth; however, for the reason assigned above, we shall estimate it at 100,000; and hence, the diameter of the earth being supposed less than 10,000 miles, the diameter of the concave firmament must be less than 1,000,000,000 miles; but for the reasons above-mentioned, we shall suppose it 1,000,000,000 miles.

(*d*) In the Latin of Clavius Milliaria. The Milliarium was equal to 8 stadia, which at 625 feet each, gives 5,000 feet for the Milliarium, less than an English mile by 280 feet.

III. That

III. That a small sphere of the size of a poppy-feed could not contain more than 10,000 of the smallest sands: This every one will concede to, since the imagination itself can scarce conceive a magnitude so small as the ten thousandth part of a poppy-feed, neither are there any particles of sand so small existing in nature. But that our determination may excite a greater degree of admiration, and more sands be comprehended in the system of the world, we shall suppose that a sphere of so small a magnitude might contain 10,000 particles of sand.

IV. That the diameter of a grain of poppy-feed is not less than the fortieth part of a geometrical inch: For according to the experiments of Archimedes (*e*), 35 grains of poppy-feed placed in a

(*e*) In all the Greek editions of the *Arenarius*, I have seen, it is 25 feeds, except the anonymous one, which has 26.—Dr. Wallis supposes that the author might have wrote 35, because Archimedes has throughout the work made use of the nearest greater number of even tens to the number found. Also in the ancient Latin and in Commandine's it is 35.

right line extended above an inch, so that in reality a poppy-seed is greater than the $\frac{1}{35}$ th part of an inch, and consequently much greater than $\frac{1}{40}$ th. However, tho' no grains of poppy are found so small, we shall, for the sake of making the demonstration still more evident, suppose the diameter of one to be only $\frac{1}{40}$ th of an inch.

V. That a mile is much less than 100,000 inches : For since four inches (*f*) make a palm, and four palms a foot, and five feet a geometrical space, and 1000 geometrical paces a mile, it is evident that 80,000 inches are equal to one mile, and consequently that it is much less than 100,000 inches : however, to make the work easier, we shall suppose a mile to consist of 100,000 inches.

Now the diameter of a poppy-seed being supposed $\frac{1}{40}$ th part of an inch (tho'

(*f*) *Digiti*.—The *Digitus* appears to have been about $\frac{72525}{100000}$ of an English inch. Sixteen of these make 11 inches and $\frac{604}{10000}$ of an inch, less than an English foot by $\frac{396}{10000}$ of an inch,

it

it is much greater) 40 grains of poppy-seeds will constitute one inch; and (since spheres are to each other in the triplicate ratio of their diameters) therefore a sphere of an inch diameter will be to one grain of poppy-feed as 64,000 is to 1: As is expressed in those four continual proportionals 1:40:1600:64000, which are to each other in the ratio of 40 to 1, or of one inch to one seed of poppy. Therefore a grain of poppy-feed being supposed to contain 10,000 particle of sands, a sphere of an inch diameter containing an equal number with 64,000 such seeds, would comprehend 64,000,0000 particles. This number is evidently much greater than the multitude of sands which could be contained in a sphere of an inch diameter, because less than 40 grains of poppy-feed make one inch, and the smallest sands are greater than the $\frac{1}{10000}$ th part of a poppy-feed: but that the work may be more expeditious and easy, we will suppose that a sphere of an inch diameter might contain not only 640,000,000 sands, but even 1,000,000,000.

Then,

Then, 100,000 inches being supposed to constitute a mile (tho' in reality a mile is much less), a sphere of a mile diameter will be to a sphere of one inch diameter as $100000 \times 100000 \times 100000$ to $1 \times 1 \times 1$ (Eucl. 18. 12.) or as 1,000,000,000,000,000 to 1; and consequently, since a sphere of an inch diameter is supposed to contain 1,000,000,000 sands, a sphere of a mile diameter being 1,000,000,000,000,000 times as large would contain 1,000,000,000,000,000,000,000,000,000 of the same particles of sand. This number is, indeed, much greater than the multitude of sands which a sphere of a mile diameter could contain; because less sands than 1000000000 would compose a sphere of an inch diameter, and fewer inches than 100000 make a mile. However, for the reasons before mentioned, we shall suppose that 1,000,000,000,000,000,000,000,000,000 sands might be contained in a sphere of a mile diameter.

Lastly, the diameter of the concave firmament being supposed 1,000,000,000 miles

* That is one thousand octillions, or the eighth degree of millions.

in

in the diameter of the concave firmament. Therefore this number last found which has 51 cyphers annexed to unity, is much greater than the number of sands which would be requisite to fill the whole expansion of space to the concave firmament of the heavens: though those sands were only equal to the $\frac{1}{100000}$ th part of a poppy-feed. From hence therefore it is evident, that we could easily determine how many sands would be requisite to fill the whole system of the world, was it known how many sands are equal to a grain of poppy-feed, and how many poppy-feeds would make one inch, and how many miles each containing 80,000 inches are contained in the diameter of the concave firmament. But as all these are as yet undetermined, we have assumed (following the example of Archimedes) the diameter of the system of the world much greater than the most skilful astronomers determine it to be; in like manner we have supposed a grain of poppy-feed equal to more sands than it really is, and an inch to comprehend more poppy-feeds

feeds than it really could. Because, that by this means a greater number might be obtained, which should far exceed (as was proposed) the number of sands which really could be comprehended within the concave superficies of the firmament of the heavens. The obtaining of which, as the sand has been generally termed innumerable, will certainly to many appear incredible.

F I N I S.



